

Generators & Rel's (§ 7.9, 7.10)Recall: S_3 sym gp

$$x = (123)$$

$$y = (12)$$

$$\begin{array}{ll} x^3 = 1 & yx = x^2y \\ y^2 = 1 & \end{array}$$

Using Rel's can "simplify" an expression in x 's & y 'seg. $y^3 x^{-5} y x \rightarrow$ int explicitly as

$$\text{an elt in } \underbrace{\{1, x, x^2, y, xy, x^2y\}}_{= S_3}.$$

Free Group Set $S = \{a, b, c, d\}$ $\mathcal{F}_S =$ free gp on S Elt's of $\mathcal{F}_S$ are words in $a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1}$

e.g. $a^3 b c a d^{-1} c^{-1} a$

$\cdot a^{-10} b c$

$\cdot 1$

$\cdot b c b c b c b c$

gp operation: concatenation of words

e.g. $(a^{-10} b c) \cdot (b c b c) = a^{-10} b c b c b c$

Rel's between symbols cancelling x w/ x^{-1} for some $x \in S$

e.g. $(a^{-10} b c) \cdot (c^{-1} b^{-1} a^5 d a)$

eg. $(a^{-10}bc) \cdot (c^{-1}b^{-1}a^5da)$

$$= a^{-10} \cancel{b} \cancel{c} b^{-1} a^5 da$$

$$= a^{-10} \cancel{b} \cancel{b^{-1}} a^5 da$$

$$= a^{-5} da$$

identity: 1 "empty word"

Thm: $\mathcal{F}_S$ is a gp. Called the "Free group on S ".

For example, clear how to make inverses:

$$\text{inverse of } abc b^{-1} c^{-5} d \text{ is } d^{-1} c^5 b c^{-1} b^{-1} a^{-1}$$

If $|S| = 1$, then $\mathcal{F}_S$ is infinite cyclic (in particular, an abelian gp)

If $|S| > 1$, then $\mathcal{F}_S$ is huge and is nonabelian.

Lemma: Suppose G a gp, $\{N_\alpha\}_{\alpha \in I}$ is a set of normal subgps

of G . Then $\bigcap_{\alpha \in I} N_\alpha = N$ is again a normal subgroup of G .

Pf: Suppose $n \in N$, $g \in G$. $gn g^{-1} \in N$.

Well $n \in N \Rightarrow n \in N_\alpha \forall \alpha$.

$$gn g^{-1} \in N_\alpha \forall \alpha \text{ b/c } N_\alpha \text{ is normal.}$$

$$\Rightarrow gn g^{-1} \in \bigcap_{\alpha \in I} N_\alpha = N. \quad \square$$

Suppose S is a set, $S = \{a, b, c, d\}$.

$\mathcal{F}_S$: the free gp on S .

Question: Can we construct a gp G that has elts a, b, c, d that satisfy the rels $\underline{a^5=1}$, $\underline{bc=cb}$, $\underline{abcd=1}$, $\underline{c^7d^2=1}$

Consider every normal subgroup of $\mathcal{F}_S$ that contains

$$\underline{a^5, bc b^{-1} c^{-1}, abcd, c^7 d^2}$$

Call these subgps

$$N_g, g \in I$$

$$N = \bigcap_{g \in I} N_g = \text{intersection of every normal subgroup that contains}$$

N is normal.

Define $G := \mathcal{F}_S / N$ the quotient gp, which we can do b/c N is normal.

Let $\underline{\bar{a}, \bar{b}, \bar{c}, \bar{d}}$ be the images of the elts a, b, c, d from $\mathcal{F}_S$ in G .
 $\bar{a} \in G$

Then $\bar{a}^5 \in G$ $\bar{a}^5 = 1$ in G b/c a^5 is in N

Similarly, $\bar{b} \bar{c} \bar{b}^{-1} \bar{c}^{-1}$ is 1 in G b/c $bc b^{-1} c^{-1}$ is in N

$$G = \langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \mid \bar{a}^5=1, bc=cb, abcd=1, c^7 d^2=1 \rangle$$

- This is a notation for the sp we just defined

- This process is called constructing a sp by generators & rels.

Example

$$G = \langle x, y \mid x^3=1, y^2=1, yx=x^2y \rangle$$

$$= \mathbb{F}_{\{x,y\}} / \langle x^3, y^2, yxy^{-1}x^{-2} \rangle$$

"smallest" normal subgroup of $\mathbb{F}_{\{x,y\}}$ that contains $x^3, y^2, yxy^{-1}x^{-2}$

G is isomorphic to S_3 .

Example · $\langle x, y \mid x^5=1, y^2=1 \rangle$: huge sp
(infinite)

· $\langle x, y \mid x^5=1, y^2=1, yx=x^3y \rangle$: finite

$$\begin{aligned} \boxed{x^3 y x y x^{-1}} &= x^3 (x^3 y) y x^{-1} \\ &= x^6 y^2 x^{-1} \\ &= x \cdot 1 \cdot x^{-1} \\ &= 1 \end{aligned}$$

Dihedral gp $n \geq 1$

$$D_n = \langle x, y \mid x^n = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$$

Turns out: • D_n has order $2n$

• $\exists \varphi: D_n \rightarrow GL_2(\mathbb{R})$ an injective homomorphism

Correspondence Thm

$\varphi: G \rightarrow G'$, surjective gp homom.

↪ relates subgps of G w/ subgps of G' .

Propⁿ $\varphi: G \rightarrow G'$ be a gp homom, w/ kernel K .

$H' \subseteq G'$ is a subgp. let $H = \varphi^{-1}(H')$ be the inverse image of H' , i.e. $H = \{h \in G : \varphi(h) \in H'\}$.

Thm • H is a subgp of G that contains K

• If H' is normal, then H is normal

• If φ is surjective and H is normal, then H' is normal.

Pf: • $H \ni K = \varphi^{-1}(1) \subseteq \varphi^{-1}(H')$

• $1 \in H$.

• $x, y \in H \Rightarrow \varphi(x) \in H', \varphi(y) \in H'$.

$\varphi(xy) = \varphi(x)\varphi(y) \in H'$ b/c H' is a subgp

$\Rightarrow xy \in H$

normal ...

$$\Rightarrow xy \in H$$

$$\bullet \quad x \in H \Rightarrow \varphi(x) \in H'$$

$$\varphi(x^{-1}) = \varphi(x)^{-1} \in H' \quad \text{b/c } H' \text{ is closed under inverses} \\ \Rightarrow x^{-1} \in H$$

Suppose

$$H' \text{ normal: } h \in H, g \in G, \quad ghg^{-1} \in H$$

$$\varphi(ghg^{-1}) = \varphi(g) \underbrace{\varphi(h)}_{\in H'} \varphi(g)^{-1} \in H' \quad \text{b/c } H' \text{ is normal}$$

$$\Rightarrow ghg^{-1} \in H \Rightarrow H \text{ is normal.}$$

Suppose φ is surjective & H is normal $\Rightarrow H'$ is normal. $\square$

Then $\varphi: G \rightarrow G'$ is surjective gp homom. w/ kernel K .

Then

$$\left\{ \begin{array}{l} \text{Subgps of } G \\ \text{containing } K \end{array} \right\} \longleftrightarrow \left\{ \text{Subgps of } G' \right\}$$

$$\left\{ \begin{array}{l} H \longmapsto \varphi(H) \\ \varphi^{-1}(H') \longleftarrow H' \end{array} \right.$$

These maps are inverse to one another and induce a bijection between the two sets above.

If H is normal, so is H' . Moreover, $|H| = |K| |H'|$.

Pf: We've essentially checked this theorem already.

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Still need to check:

- $\varphi^{-1}(\varphi(H)) = H$

- $\varphi(\varphi^{-1}(H')) = H'$

• $\varphi(\varphi^{-1}(H')) = H'$: True for any surjective map of sets φ

• $\varphi^{-1}(\varphi(H)) = H$: Clear: $H \subseteq \varphi^{-1}(\varphi(H))$

Conversely, must check: $\varphi^{-1}(\varphi(H)) \subseteq H$.

$$\text{Suppose } x \in \varphi^{-1}(\varphi(H)) \Rightarrow \varphi(x) \in \varphi(H)$$

$$\Rightarrow \exists h \in H \text{ s.t. } \varphi(x) = \varphi(h).$$

$$\Rightarrow \varphi(h^{-1}x) = 1 \Rightarrow h^{-1}x \in K.$$

$$\text{But } K \subseteq H \Rightarrow h^{-1}x \in H \Rightarrow x \in hH = H$$